Time quantization and *q*-deformations

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Abstract
We extend to quantum mechanics the technique of stochastic subordination, by means of which one can express any semi-martingale as a time-changed Brownian motion. As examples, we considered two versions of the q-deformed harmonic oscillator in both ordinary and imaginary time and show how these various cases can be understood as different patterns of time quantization rules.

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1. Introduction

In a search to unravel the fabric of space at short distances, many authors have explored variations on ordinary quantum mechanics based on q-deformations of the canonical commutation relation, q being a parameter in the interval $(0, 1)$ where 1 corresponds to the Bose limit, see for instance [1, 2, 4–10]. Time quantization was also considered, see [3, 13]. On an entirely different line of research, probabilists developed the notion of stochastic time changes (also called stochastic subordination) as a way of understanding jump processes, see [11, 12, 14]. This work gave rise to a representation of Levy processes, a family of translation invariant jump processes, as subordinated Brownian motions whereby the time change is uncorrelated with the underlying process. More generally, Monroe proved that all semi-martingales can be represented as time-changed Brownian motions, as long as one allows for the subordinator to be correlated, see [15].

In this paper we bring together ideas from all these lines of research and show that one can interpret q-deformations in terms of stochastic time changes, albeit of a new type which is designed in such a way to preserve quantum probability. We find that these representations provide new insights into the notion of q-deformation and indicate an alternative, physically intuitive path to understand short-scale deformations of quantum field theory.

2. Stochastic subordination

Stochastic subordination is a procedure for constructing a stochastic process from another by means of a stochastic time change. If X_t is a stochastic process, the subordinated process $\tilde{X}_t$ is defined as follows:

$$\tilde{X}_t = X_{T_t} \tag{1}$$

where T_t is a monotonically non-decreasing process. The process T_t is called the Bochner subordinator in case its increments $T_{t+\Delta t} - T_t$ are independent and their distribution depends only on the time elapsed Δt but not on t. Such a time-homogeneity property makes the Bochner subordinator also potentially appealing for basic quantum physics. Under such hypothesis, if the process X_t is stationary and Markov and G is its generator, then the process $\tilde{X}_t$ is also a stationary Markov process. One can show under mild conditions that the generator $\tilde{G}$ of $\tilde{X}_t$ can be expressed as a function $\tilde{G} = -\phi(-G)$. It is useful to sketch a quick proof in order to work out a quantum extension.

Suppose that the generator G admits a complete set of eigenfunctions f_n with eigenvalues $\lambda_n \leqslant 0$. (The argument below can readily be extended to the case where G also has a continuous spectrum.) We have that

$$\rho_t(x) = \sum_{n=0}^{\infty} a_n \, e^{\lambda_n t} f_n(x) \tag{2}$$

where $\rho_0(x)$ is a fixed initial condition. If one instead evolves the initial state according to the $\tilde{X}_t$ dynamics the distribution $\tilde{\rho}_t(x)$ is given by

$$\tilde{\rho}_t(x) = \sum_{n=0}^{\infty} a_n \left[\int e^{\lambda_n s} \mu_t(\mathrm{d}s) \right] f_n(x). \tag{3}$$

Hence, there ought to be a function ϕ such that

$$\int e^{\lambda s} \mu_t(\mathrm{d}s) = e^{-t\phi(-\lambda)} \tag{4}$$

an equation that needs to hold true for all times t and all values of $\lambda \leqslant 0$. This equation constrains the form of the functions ϕ as not all functions admit a one-parameter family of positive measures $\mu_t(\mathrm{d}s)$ such that the above equation is satisfied. The measures $\mu_t(\mathrm{d}s)$, if well defined, are called the renewal measures [11].

Note that the Bochner subordination also applies to general positivity preserving contraction semigroups R_t, as the subordinated semigroup $\tilde{R}_t$ can be consistently defined as follows:

$$\tilde{R}_t \rho_0 = \int R_s \rho_0 \, \mu_t(\mathrm{d}s). \tag{5}$$

3. Quantum deformations and quantum subordination

In the quantum mechanics case the problem is different as conservation of quantum probability requires that the dynamics be defined by one-parameter groups of unitary transformations. If $\mathbb{H}$ is a quantum Hamiltonian and $\mathbb{U}_t = e^{-it\mathbb{H}}$ is the corresponding dynamics, the subordinated dynamics is defined as follows:

$$\tilde{\mathbb{U}}_t = \int \mathbb{U}_s \chi_t(\mathrm{d}s). \tag{6}$$

Consistency with quantum probability conservation requires that $\chi_t(\mathrm{d}s)$ be a complex valued measure such that

$$\int \mathrm{e}^{-\mathrm{i}\lambda s} \chi_t(\mathrm{d}s) = \mathrm{e}^{-\mathrm{i}t\phi(\lambda)} \tag{7}$$

for some real valued function $\phi(\lambda)$. This condition restricts the form of the measures $\chi_t(\mathrm{d}s)$. As extensions to the renewal measures, we shall call $\chi_t(\mathrm{d}s)$ the *quantum renewal measures*.

In the following we consider several examples using two versions of the q-deformed harmonic oscillator as an example. A first example is provided by the one-mode harmonic oscillator Hamiltonian

$$\mathbb{H} = \frac{\hbar\omega}{2}\left(a_q^\dagger a_q + a_q a_q^\dagger\right) \tag{8}$$

where the creation and annihilation operators a_q and $a_q^\dagger$ satisfy the following relations:

$$a_q a_q^\dagger - q a_q^\dagger a_q = 1. \tag{9}$$

This deformation scheme was proposed by Arik and Coon in [1] and it leads to the energy spectrum

$$\epsilon_q(n) = \frac{\hbar\omega}{2}\frac{1-q^n}{1-q}. \tag{10}$$

This means that the q-deformed Hamiltonian operator $\tilde{\mathbb{H}}$ can be represented so that

$$\tilde{\mathbb{H}} = \phi(\mathbb{H}) \equiv \frac{1-q^{\mathbb{H}}}{1-q}. \tag{11}$$

In the Euclidean picture, one considers the semigroup $\mathrm{e}^{-t\tilde{\mathbb{H}}}$, which can be represented as follows:

$$\mathrm{e}^{-t\tilde{\mathbb{H}}} = \mathrm{e}^{-\frac{t}{1-q}} \sum_{n=0}^{\infty} \frac{t^n}{(1-q)^n n!} \mathrm{e}^{-n|\ln q|\mathbb{H}} = \int \mathrm{e}^{-s\mathbb{H}} \mu_t(\mathrm{d}s) \tag{12}$$

where

$$\mu_t(\mathrm{d}s) = \mathrm{e}^{-\frac{t}{\delta t}} \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{t}{\delta t}\right)^n \delta(s - n|\ln(1-\delta t)|)\,\mathrm{d}s. \tag{13}$$

Note that $\mu_t(\mathrm{d}s)$ is the distribution at time t of a Poisson process of characteristic time δt.

In the real time, quantum picture, we have to account for phases required to maintain quantum probability conservation, namely

$$\mathrm{e}^{-\mathrm{i}t\tilde{\mathbb{H}}} = \mathrm{e}^{-\frac{\mathrm{i}t}{1-q}} \sum_{n=0}^{\infty} \frac{(\mathrm{i}t)^n}{(1-q)^n n!} \mathrm{e}^{-n|\ln q|\mathbb{H}} = \int \mathrm{e}^{-\mathrm{i}s\mathbb{H}} \chi_t(\mathrm{d}s) \tag{14}$$

where the quantum renewal measure characterizing the quantum subordinator is given by

$$\chi_t(\mathrm{d}s) = \frac{\mathrm{e}^{-\frac{\mathrm{i}t}{\delta t}}}{\pi\mathrm{i}} \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\mathrm{i}t}{\delta t}\right)^n \frac{1}{s + \mathrm{i}n|\ln(1-\delta t)| + \mathrm{i}0}\,\mathrm{d}s. \tag{15}$$

By reusing the terminology in [3], the location of the poles of the quantum renewal measure will be referred to as 'chronons', or time quanta.

In summary, these expressions show that the Arik–Coon q-deformation scheme for the harmonic oscillator is equivalent in the Euclidean picture to a time quantization, whereby the imaginary time proceeds according to a Poisson process and has increments equal to discrete quanta of $\delta t = 1 - q$. In the real, quantum mechanics time picture, a similar Poisson
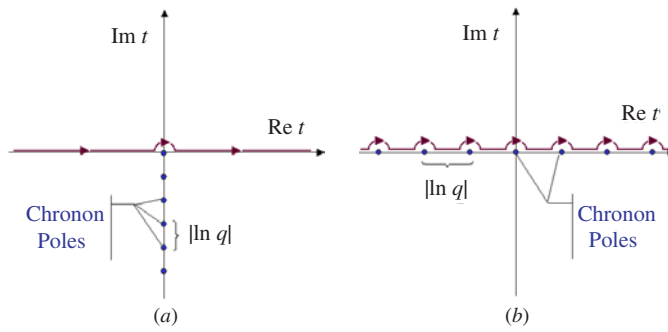
Figure 1. Quantized time representations for two q-deformation schemes. (a) Arik–Coon scheme. (b) Biedenharn–Macfarlane scheme.

distribution applies except that additional phases have to be added to the expansion in such a way to ensure conservation of quantum probability. In the latter representation, chronons are located at equally spaced intervals along the imaginary axis.

Alternative deformation rules have been proposed by Biedenharn in [2] and MacFarlane in [4]. According to this scheme q is a parameter which can be not only real, but can also take values in the unit circle $S^1 = \{e^{i\alpha}, \alpha \in [0, 2\pi)\}$. According to this scheme the creation and annihilation operators a_q and $a_q^\dagger$ satisfy the following relations:

$$a_q a_q^\dagger - q a_q^\dagger a_q = q^{-n} \tag{16}$$

and the energy spectrum is

$$\epsilon_q(n) = \frac{\hbar\omega}{2} \frac{q^n - q^{-n}}{q - q^{-1}}. \tag{17}$$

In the following, we shall indeed assume that $q = e^{i\alpha}$, so that the q-deformed Hamiltonian operator $\tilde{\mathbb{H}}$ is thus

$$\tilde{\mathbb{H}} = \phi(\mathbb{H}) \equiv \frac{q^{\mathbb{H}} - q^{-\mathbb{H}}}{q - q^{-1}} = \frac{\sin\alpha\mathbb{H}}{\sin\alpha}. \tag{18}$$

The function ϕ does not correspond to a Bochner subordinator, as there is no one-parameter family of positive measures such that relation (4) holds . Hence in the Euclidean picture, one cannot consistently construct a positivity preserving semigroup. In real time however, there is no such restriction on the measure and we find

$$e^{-it\tilde{\mathbb{H}}} = \sum_{n=0}^{\infty} \left(\frac{-t}{2\sin\alpha}\right)^n \sum_{k=0}^{n} \frac{(-1)^k}{k!(n-k)!} e^{i\alpha(n-2k)\mathbb{H}} = \int e^{-is\mathbb{H}} \chi_t(\mathrm{d}s) \tag{19}$$

where the measure characterizing the quantum subordinator is given by

$$\chi_t(\mathrm{d}s) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \left(\frac{-t}{2\sin\alpha}\right)^n \sum_{k=0}^{n} \frac{(-1)^k}{k!(n-k)!} \frac{1}{s + (n-2k)\alpha + i0} \, \mathrm{d}s. \tag{20}$$

In this case, time is quantized along the real axis and the chronons, or quantized time values, are poles of the measure $\chi_t(\mathrm{d}s)$. The time quanta, or the chronon increment, are given by $\alpha = |\ln q|$, as in the Arik–Coon q-deformation scheme. It is instructive to compare both q-deformation schemes and plot the distribution of the respective chronons. Figure 1 illustrates furthermore the contour in the lower half-plane, used to integrate the measures $\chi_t(\mathrm{d}s)$ in (15) and (20).

4. Conclusions

Stochastic subordination is one of the most common methodologies to deform stochastic processes. In this paper, we introduce an analogue concept of quantum subordination that is appropriate to deform quantum mechanical systems. We show that this notion is broad enough to capture two examples that received much attention in the literature, namely two versions of the q-deformed harmonic oscillator. In these two cases, we show how quantum subordination is defined by means of a quantum renewal function, an analytic function with poles corresponding to time quanta or, to reuse the terminology in [3], chronons. These concepts provide a new angle to the notion of q-deformation and shed light on a physically intuitive route to model short-scale deformations of quantum field theory.

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